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On the determination of the rail support modulus k

Arnold D. Kerr

Department of Civil & Environmental Engineering, University of Delaware, DuPont Hall, Newark, DE 19716-3120, USA

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Abstract

With the adoption and use of equation $EIw^{\mathrm{IV}} + kw = q$ for track analyses, various methods for the determination of k, the rail support modulus, were proposed during the past several decades. However, some of these methods are difficult to use or are of questionable validity. Therefore, at first, a simple yet accurate method that is consistent with this equation is presented and its use is illustrated on practical examples. Then, other published methods are critically reviewed and their shortcomings are pointed out. © 2000 Elsevier Science Ltd. All rights reserved.

Keywords: Railways; Track modulus; Continuously supported structures

1. Introduction

The standard analysis of cross-tie tracks assumes that the rail responds like an elastic beam that is attached to a continuous base of closely spaced elastic springs. The corresponding governing equation is

$$EI\frac{\mathrm{d}^4 w}{\mathrm{d}x^4} + kw(x) = q(x) \tag{1}$$

in which $w(x)$ is the vertical deflection of the rail axis at point x, EI is the vertical flexural stiffness of the rail, $q(x)$ is the vertical load caused by the wheels, and

$$kw(x) = p(x) \tag{2}$$

is the contact pressure between the rail and its base. k is denoted as the rail support modulus or track modulus. Eq. (1) is recommended in the AREA Manual (1991, Section 22, Part 3).

For one wheel load of magnitude P, shown in Fig. 1, the deflection curve is obtained as

E-mail address: adkerr@ce.udel.edu (A.D. Kerr).

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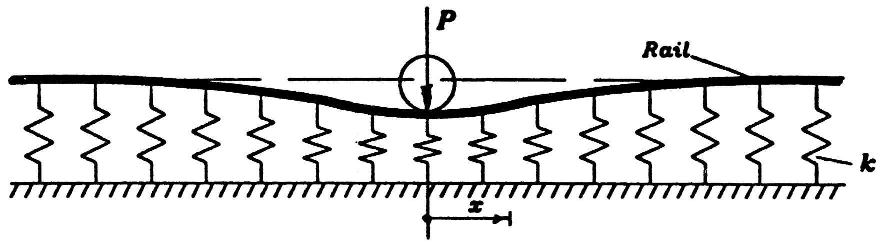

Fig. 1. Rail subjected to a wheel load.

$$w(x) = \frac{P\beta}{2k} e^{-\beta x} (\cos \beta x + \sin \beta x) \tag{3}$$

where

$$\beta = \sqrt[4]{\frac{k}{4EI}}. \tag{4}$$

For more than one load, the rail deflection may be obtained by superposing the effects of the various wheel loads.

For the analysis of a track structure, the parameters that enter expression (3) are needed. E is Young's modulus of rail steel and is known, I is the bending moment of inertia of the rail under consideration and is listed in the AREA Manual (1991, Chapter 4), and P is a known wheel load. The only unknown is the track modulus k.

2. An early method for the determination of *k*

In this approach, k is determined by equating a measured rail deflection at one point with the corresponding analytical expression based on Eq. (3). For several decades, promoted by the research of Timoshenko and Langer (1932), the used loading device consisted of one axle, as shown in Fig. 2. In this procedure the rail deflection at the wheel, w_{m}, caused by one wheel load P, is recorded and then collocated (i.e. equated) with the corresponding analytical expression obtained from Eq. (3) at $x = 0$; namely, by setting $w_{\mathrm{m}} = w(0)$ in Eq. (3). The resulting equation is

$$w_{\mathrm{m}} = \frac{P\beta}{2k} = \frac{P\sqrt[4]{\frac{k}{4EI}}}{2k}. \tag{5}$$

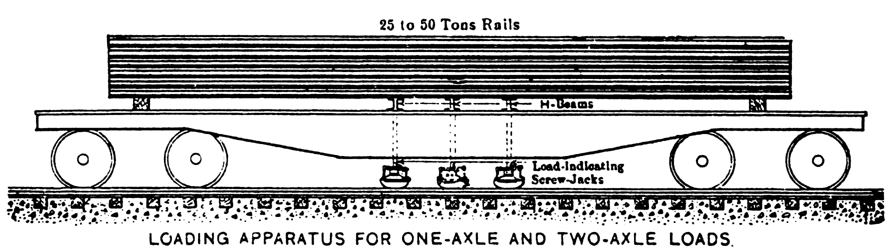

Fig. 2. Talbot committee loading device (Talbot Committee, 1918).

Solving it for k, the only unknown, yields

$$k = \frac{1}{4}\sqrt[3]{\frac{P^4}{EI\,w_m^4}};\tag{6}$$

an explicit expression for the rail support modulus.

As an example consider a track with 136 RE rails loaded by one axle with a wheel load $P = 30{,}000$ lb. The recorded deflection caused by P is $w_m = 0.12$ in. According to Eq. (6) the corresponding rail support modulus (i.e. track modulus for one rail) is

$$k = \frac{1}{4}\sqrt[3]{\frac{(30{,}000)^4}{30\times10^6\times95.00\times(0.12)^4}} = 2775\ \left(\mathrm{lb/in.^2}\right).\tag{7}$$

The above method is very simple, since it requires only one deflection measurement and a simple calculation. Another advantage is that because of the bending stiffness of the rail the ballast-subgrade conditions are averaged out over the affected track section. Because of its simplicity, Eq. (6) is being recommended for the calculation of the track modulus k, even in the recently published texts on railroad engineering. For examples refer to work by Hay (1982, p. 262) and to Eisenmann (in Fastenrath, 1981, p. 36).

The major shortcoming of using Eq. (6) for the determination of the track modulus k is that it requires a special test set-up with one-axle wheel loads. One such set-up was used by the Talbot Committee (1918) for determining rail deflection profiles. It consisted of a flat car loaded with rails weighing 25–50 tons, and equipped with load indicating screw jacks, as shown in Fig. 2.

The outer jacks were used to simulate two-axle loadings of a truck, whereas the middle one simulated a one-axle load. Cars of the same type have been used for the determination of track modulus in western Europe (Driessen, 1937; Birmann, 1957; Nagel, 1961) and in the former USSR (Kuptsov, 1975) to simulate a one-axle load. A static one-axle loading device was also used by Zarembski and Choros (1980) in the AAR laboratory in Chicago. But such special one-axle loading devices are, generally, not available to railway engineers; or for that matter, not even to the majority of railway researchers.

It was therefore essential to establish a procedure that retains the simplicity of the above method, but is able to utilize any available car or locomotive on two or three-axle trucks as a loading device. This was done by Kerr (1983, 1987).

3. The Kerr method for determination of k using any available car or locomotive

To demonstrate this method, consider a car on two-axle trucks, as shown in the insert of Fig. 3. The analytical expression for the rail deflection at the left wheel of truck (I) is obtained by superposition, using Eq. (3). Since all wheel loads of a truck are equal, but the load exerted by each truck may be different, we set

$$P_1 = P_2 = P \quad \text{and} \quad P_3 = P_4 = nP\tag{8}$$

where n is known. The number n is obtained by weighing; namely by placing the truck (I) and then truck (II) on a track scale.

The analytical expression for the vertical rail deflection at the left wheel of truck (I), caused by all four wheels of the two trucks, is obtained by superposing the corresponding $w(x)$ expressions given in Eq. (3). It is, since $l_1 = 0$,

$$w(0) = \frac{P\beta}{2k} + \frac{P\beta}{2k}e^{-\beta l_2}(\cos\beta l_2 + \sin\beta l_2) + \frac{nP\beta}{2k}e^{-\beta l_3}(\cos\beta l_3 + \sin\beta l_3) + \frac{nP\beta}{2k}e^{-\beta l_4}(\cos\beta l_4 + \sin\beta l_4), \tag{9}$$

where

$$\beta = \sqrt[4]{\frac{k}{4EI}}. \tag{10}$$

The rail support modulus k is obtained by collocating (equating) this deflection with the deflection measured at the left wheel, w_m; namely $w(0) = w_m$. This yields

$$\frac{w_m}{P} = \frac{\beta}{2k}\left[1 + e^{-\beta l_2}(\cos\beta l_2 + \sin\beta l_2) + ne^{-\beta l_3}(\cos\beta l_3 + \sin\beta l_3) + ne^{-\beta l_4}(\cos\beta l_4 + \sin\beta l_4)\right]. \tag{11}$$

In above equation all quantities, except k, are known for a given field test. This equation is equivalent to Eq. (5) for one wheel load. Whereas Eq. (5) was solved explicitly for k, this is not possible for Eq. (11).

To avoid involved solutions of the above transcendental equation for k, the right-hand side of Eq. (11) was evaluated numerically for given sets of (E, I, w_m, P) values by substituting different values of k, from the range $500 \leq k \leq 9000$ lb/in.^{2} It was assumed that the wheel distances are those of a freight car with $l_1 = 0$, $l_2 = 5$ ft 10 in. = 70 in., $l_3 = 46$ ft 3 in., and $l_4 = 52$ ft 1 in. (distance between truck centers is 46 ft 3 in.). The results of this numerical evaluation are presented graphically in Fig. 3.

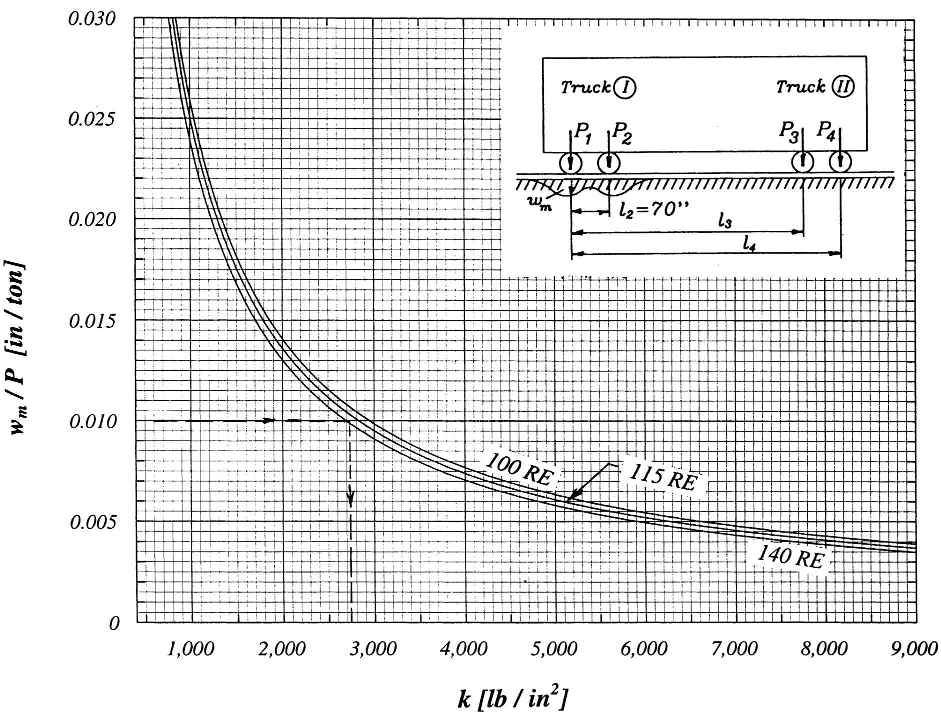

Fig. 3. Master chart for the determination of k using a vehicle on 2-axle trucks.

To check the effect of truck (II) on the results, the evaluations were conducted for $n = 1.0$, 0.5, and 0. It was found that for the wheel set distances l_3 and l_4 used, truck (II) had no noticeable effect on the w_m/P values, even for k as low as 1000 lb/in.[2] Thus, whether the wheel loads of truck (II) are equal to those of truck (I) (i.e. $n = 1$) or they are only about one half of those of truck (I) (i.e. $n = 0.5$), the curves presented in Fig. 3 are still valid. This is a useful finding, since the vertical forces a loaded car, or a locomotive, exert on their trucks generally differ.

The graphs presented in Fig. 3 are for rails 100, 115, and 140 RE. Those for other rail sizes were not included due to space limitation between the shown curves. However, because of the proximity of the presented curves, values for the missing rail sizes may be easily obtained by interpolation. The same argument applies to worn rails.

It is proposed to use the graphs in Fig. 3 for the determination of the k modulus, as follows: First measure the deflection w_m caused by a car on two-axle trucks with wheel loads P_m, as shown in the insert of Fig. 3. Then form w_m/P_m. The graph for the corresponding rail yields directly the k-value.

As an example, a loaded freight car on two-axle trucks is chosen as a loading device for the field test to be performed, at a track location of interest. As a first step, the wheel loads of one of the trucks, say truck (I), are determined by placing the truck on a car scale for weighing. It was found to be 118,400 lb. Assuming that each of the four wheels in truck carries approximately the same load, the wheel load P_m is calculated as

$$P_m = 118{,}400/4 = 29{,}600 \text{ lb} = 14.8 \text{ tons}. \tag{12}$$

Next, a fine scale equipped with a magnet is attached vertically to the rail web, at the track location of interest. Then the test car is moved to this location. When the front wheel of truck (I) reaches the point above the scale, the vertical rail deflection is recorded using a level placed about 30 ft from the rail; say $w_m = 0.15$ in. The ratio w_m/P_m is then formed; namely

$$\frac{w_m}{P_m} = \frac{0.15}{14.8} = 0.0101 \text{ in./ton}. \tag{13}$$

The test was conducted on a track with 132 RE rails that showed minor wear. With $w_m/P_m = 0.0101$ in./ton, the graphs in Fig. 3 yield directly

$$k \cong 2730 \text{ lb/in.}^2 \tag{14}$$

This completes the determination of k at this location. Note, that by using the graphs in Fig. 3, the track modulus k is obtained for a given w_m/P_m-value without any additional calculations.

To determine the k-value at another location, move the fine scale and then the test car to the new location, measure w_m, calculate w_m/P_m, and get the corresponding k-value from Fig. 3.

The procedure for determining the track modulus using a locomotive on two-axle trucks is the same as the one discussed above, except that Eq. (11) has to be evaluated for different values of the axle spaces l_2, l_3, l_4, if the wheel loads of truck (I) are the same.

The graphs in Fig. 3 exhibit an interesting feature. When formulating the governing Eq. (1), it was assumed that the k-value (the stiffness of the elastic spring layer) represents the response of the base under the rail, thus, of the cross-ties, fasteners, tie-pads, ballast and subgrade; but not the rail response. However, according to Fig. 3, as well as Eq. (6), k does depend on the rail size; although this dependence is very small.

In situations when a locomotive or car on three-axle trucks is to be used as a test vehicle, Eq. (11) has to be expanded, by including the effect of the additional axles. Denoting the wheel loads, shown in Fig. 4, as

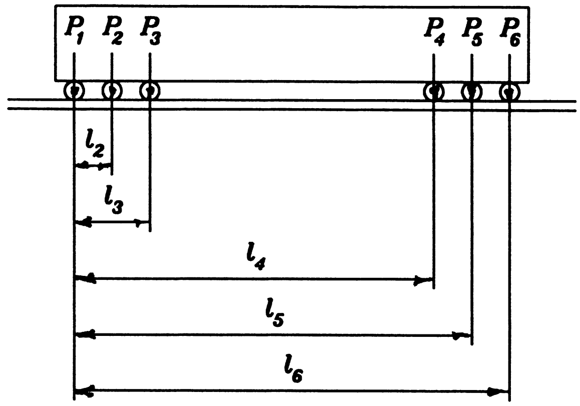

Fig. 4. Car or loc on 3-axle trucks.

$$\begin{aligned} &P_1 = P \quad\quad P_2 = n_2 P \quad P_3 = n_3 P \\ &P_4 = n_4 P \quad P_5 = n_5 P \quad P_6 = n_6 P \end{aligned} \tag{15}$$

where the $n_2, \ldots, n_6$ values are obtained by using a weighing scale and setting $w(0) = w_{\mathrm{m}}$, the formula that corresponds to Eq. (11) becomes

$$\begin{aligned} \frac{w_{\mathrm{m}}}{P} = \frac{\beta}{2k}\big[&1 + n_2 e^{-\beta l_2}(\cos \beta l_2 + \sin \beta l_2) + n_3 e^{-\beta l_3}(\cos \beta l_3 + \sin \beta l_3) + n_4 e^{-\beta l_4}(\cos \beta l_4 + \sin \beta l_4) \\ &+ n_5 e^{-\beta l_5}(\cos \beta l_5 + \sin \beta l_5) + n_6 e^{-\beta l_6}(\cos \beta l_6 + \sin \beta l_6)\big], \end{aligned} \tag{16}$$

noting again that $l_1 = 0$.

Next, the above equation has to be evaluated numerically for various rail sizes and a range of k-values, as done previously with Eq. (11). The results of this evaluation are to be plotted as graphs in a master chart, similar to the one shown in Fig. 3. The procedure for determining the rail support modulus k is as before; first roll the test car on three-axle trucks to the location of interest, next measure the vertical deflection w_{m} at the first wheel with load $P_1 = P$ shown in Fig. 4, then form w_{m}/P and get the k-value from the corresponding graph.

Note, that Eq. 16 was derived for the case when the wheel deflection is measured at the first or the last wheel of the locomotive or car. Should it be planned instead to record deflections at any of the other wheels, then Eq. 16 has to be modified accordingly.

From the above presentation it follows that for the determination of the rail support modulus k, any car or locomotive may be utilized as a loading device and that only one measured rail deflection, w_{m}, is required. The proposed method avoids the numerical solution of the involved transcendental Eq. (11) or (16) for the unknown k. It requires only the numerical evaluation of the right-hand side of the corresponding equation for various k values, which may be easily performed even on a programmable pocket calculator. The mobility of the chosen car or locomotive and the simplicity of determining the rail support modulus from one measured deflection w_{m} and a graph of the type shown in Fig. 3, allows for a rapid and economical determination of the rail support modulus k (i.e. track modulus) at various track locations.

4. Other proposed methods for determination of *k*

At this stage, it is instructive and useful to discuss other methods for the determination of k that were proposed in the railroad literature, but are of questionable validity.

In one of these methods, the field test consisted of loading vertically only one tie that was separated from the rails by removing the fasteners, then by recording the vertical displacement of this tie, and by calculating the base parameter using Eq. (2) under the assumption that the tie-ballast pressure is uniform. In one test series, the loads were generated by a freight car of about 16 tons, which was equipped with two hydraulic jacks (one at each rail seat); a similar set-up to the one shown in Fig. 2. The jacks, when activated, pressed against the tie, lifting up the car; thus, exerting about 8 tons on each rail-seat. According to Driessen (1937), 385 tests of this type were conducted before World War II on the German, Dutch, and Swiss railroads for the purpose of determining the corresponding k-values. This effort was not successful, because it did not yield meaningful results. It is worth noting that tests of this type were conducted by the German railways (DB) also after 1945, as described by Birmann (1957) and Nagel (1961).

It appears that the main problem with this method was that the used test, that loaded only one tie, has two major shortcomings. The first one is that because of the granular nature of the ballast and subgrade, their material properties may strongly vary along the track. Thus, the loading of one tie, at different locations along the track, will necessarily show a wide scatter in the obtained data. This is very apparent from the test data presented by Driessen (1937, p. 123). The second shortcoming is that the base parameter k, that is a property of a layer of closely spaced individual springs, depends on the size of the loading area when used for a continuous base consisting of ballast and subgrade (For a recent proof of this assertion refer to Kerr (1987, p. 40)). Thus, the test that uses only one tie will not yield the same parameter k as when loading a row of closely spaced ties encountered in an actual track. In this connection note that according to Wasiutynski (1937), the k-value obtained when loading only one tie is about twice as large as when using the actual rail-tie structure. The above discussion suggests that, for the determination of k, the use of tests that load only one separated tie should be avoided.

Another method for the determination of k was proposed and used by the Talbot Committee (1918) and by Wasiutynski (1937). In this method a car is moved to the track location of interest, and the caused vertical rail deflections at each tie are measured, as shown in Fig. 5. According to the Talbot Committee (1918) the rail support modulus k is then calculated by dividing the sum of the wheel loads ΣP that act on one rail, by the area formed between the undeformed straight rail and the deflected rail, A_R.

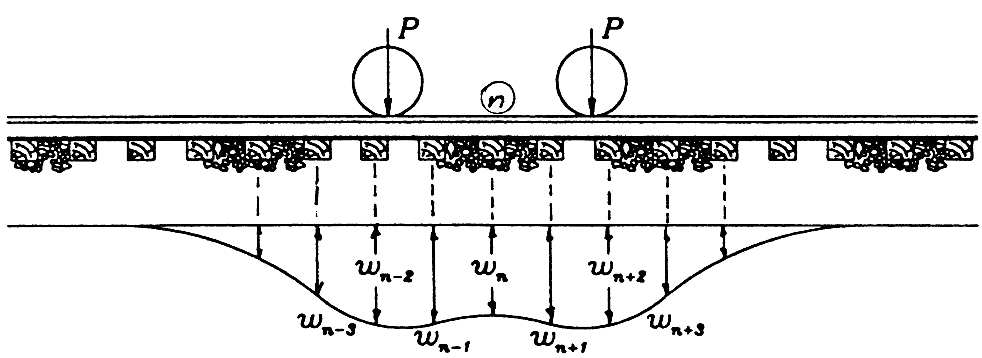

Fig. 5. Recorded rail deflections in depressed region.

This prescription for the determination of k may be derived from vertical equilibrium of a rail. Noting that $p(x)$ is the pressure that acts on the rail base (positive upwards), it follows that

$$\Sigma P - \int_{-\infty}^{\infty} p(x)\,\mathrm{d}x = 0. \tag{17}$$

Noting that $p(x) = kw(x)$, where k is constant along the track, and by definition is valid for one rail only, above equation becomes

$$\Sigma P - k\int_{-\infty}^{\infty} w(x)\,\mathrm{d}x = 0. \tag{18}$$

Solving for k we obtain

$$k = \frac{\Sigma P}{\int_{-\infty}^{\infty} w(x)\,\mathrm{d}x}. \tag{19}$$

Since the integral in the denominator is the area formed by the deflected rail, A_R, the above k-expression proves that the prescription by the Talbot Committee satisfies vertical equilibrium.

However, already early tests conducted by the Talbot Committee (1918, Chapter IV) revealed that the vertical rail deflections were not increasing linearly with increasing wheel loads, especially for tracks in poor condition, since the base stiffens. A similar type of nonlinear response was recorded more recently by Zarembski and Choros (1980), for track in good condition but for larger wheel loads.

The observed nonlinearity for relatively light wheel loads was attributed mainly to the play between the rails and the ties, the play between the ties and ballast, and the bending of the ties while they take full bearing in the ballast. For heavy wheel loads, an additional contributor to the nonlinear response is the stiffening of the track caused by the increasing compression of the ballast and subgrade layers.

To take into consideration this nonlinearity, in a later paper the Talbot Committee (1933, Chapter 37) recommended to retain the linear analysis based on Eq. (1), but to determine the rail support modulus, k, using the difference between the vertical deflections from a heavy and a light car; thus, using the reduced shaded area shown in Fig. 6. For the determination of k, they proposed the formula

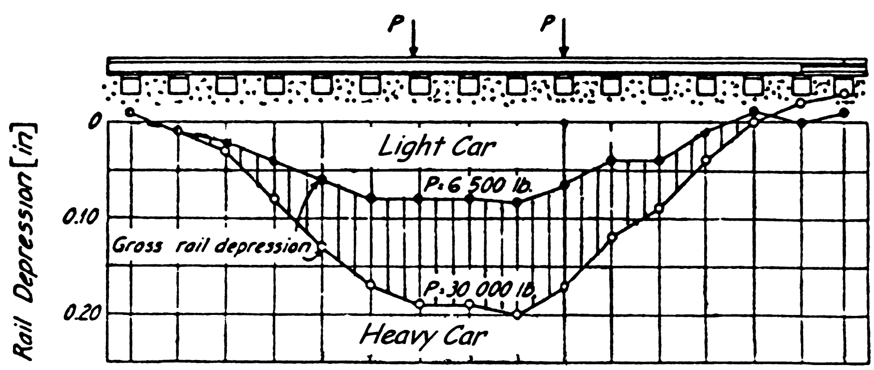

Fig. 6. Reduced deflection area.

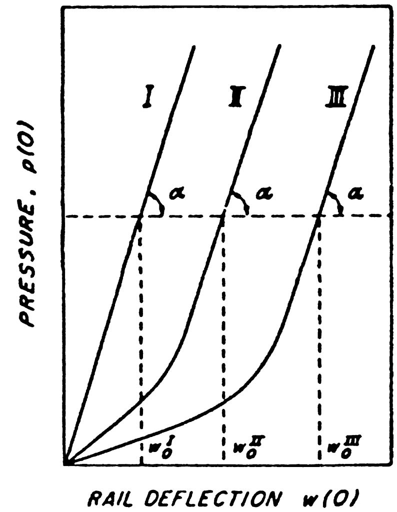

Fig. 7. Contact pressure vs. rail deflections.

$$k = \frac{\Sigma(P_{\rm h} - P_{\rm l})}{a\Sigma_{i=1}^{m}\left(w_i^{\rm h} - w_i^{\rm l}\right)} \tag{20}$$

where a is the center-to-center tie spacing, h corresponds to heavy and l to light wheels and deflections.

The justification of this assumption was that the light wheel loads will eliminate the slack at all ties in the depressed track region and that further rail deflections, beyond those caused by the light wheel loads, will be proportional to the additional loads generated by the heavy wheels. For additional details refer to Kerr and Shenton (1985). This method was used since then by many railway engineers and researchers. However it is conceptually incorrect, as explained next.

To demonstrate this point consider, as example, the three curves of rail-tie contact pressure vs. rail deflection at a point $x = 0$, as shown in Fig. 7. First, assume that each rail is pre-loaded by a uniformly distributed vertical load, as indicated by the horizontal dashed line. The corresponding vertical displacements for each rail are uniform, but they differ in magnitude for each of the cases I, II, and III. Thus, in all three cases no bending moments are generated in the rails. Then, each rail is subjected additionally to a wheel load P. Each rail will respond linearly with $k = \tan\alpha$, and the rail deflections and bending moments caused by this additional load P, will be the same for all three cases.

However, when each rail is subjected only to a heavy wheel load P (without a large uniform pre-loading) the resulting deflections and bending moments will strongly differ from the ones described above. This was shown analytically by Kerr and Shenton (1986). Thus, when considering rails subjected to wheel loads whose base exhibits a nonlinear load-deflection response as shown in Fig. 7, a situation encountered especially on North American freight lines, the 'soft' part of this response should not be neglected; otherwise, the determined rail support modulus will be too high.[1]

Since the 'reduced area' method described above requires many rail deflection measurements, recently Selig and Li (1994) proposed to simplify the determination of k by conducting a test that uses a single increasing wheel load and generates a load-deflection curve for one point, of type III in Fig. 7. They

[1] This comment also applies to the tie-paid test as specified in AREA Manual (1993, Chapter 1.9.1.15c).

proposed to determine the track modulus as $k = \tan\alpha$, where α is the angle of the steep part. This method for the determination of k is not correct either, since it neglects the 'soft' part of the response curve, and thus will result in a k-value that is too high.

In still another approach to determine the rail support modulus k, various researchers in North America and Europe assumed that the rail supporting base (consisting of pads, ties, ballast, and subgrade) may be represented by layers of springs each with a different stiffness, arranged in series, as shown schematically in Fig. 8. The resulting rail support modulus for the entire base is

$$k = \frac{1}{1/k_\mathrm{p} + 1/k_\mathrm{t} + 1/k_\mathrm{b} + 1/k_\mathrm{s}}, \tag{21}$$

where k_p is the corresponding stiffness of the pad (if used), k_t is the stiffness of the tie (due to the compressibility of wood in the rail-seat region and tie bending), k_b is the vertical stiffness of the ballast layer, and k_s is the stiffness of the subgrade. For a discussion of this method refer to Novichkov (1955), Luber (1962), Shchepotin (1964), Shakhunyants (1965), Birmann (1965/66) and Ahlbeck et al. (1978, pp. 242–243).

This approach, although intuitively appealing, is not practical for the determination of k, because the correlation of the response of a sample of (disturbed) ballast or subgrade tested in a lab with the corresponding k_b or k_s value for an actual track is not very reliable. Also, the ballast and subgrade properties generally vary along the track and additionally the response of each layer may be non-linear. Therefore, it appears that this approach is not suitable for the determination of the k-values for actual tracks.

Finally, it is instructive to discuss the method for the determination of the rail support modulus, used extensively in the *German language railroad literature*. For examples of this approach refer to the books by Hanker (1952, Chapter V.3.d), Schoen (1967, p. 263), Eisenmann (in Fastenrath, 1981, Part 2, Section 3.1) and Führer (1978, Section 3.1.2.1).

Their approach is based on the original assumption by Winkler (Winkler, 1867, Section 195) for the longitudinal-tie track, that the contact pressure between tie and support is

$$p^*(x) = Cw(x), \tag{22}$$

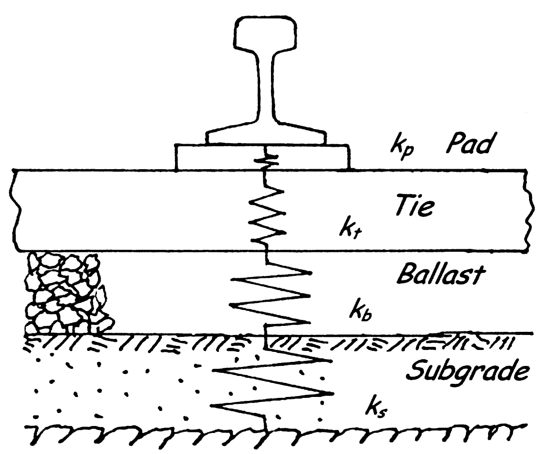

Fig. 8. Track model consisting of spring layers.

where p^* has the dimension of force per unit area, and C is the base parameter that is independent of the tie width. Since, in the differential equation for a continuously supported beam

$$EI\frac{d^4w}{dx^4} + p(x) = q(x) \tag{23}$$

each term, including $p(x)$, is of the dimension force per unit length, Winkler formed

$$p(x) = b_0p^*(x) = b_0Cw(x), \tag{24}$$

where b_0 is the width of the longitudinal-tie. Substituting it into Eq. (23), Winkler obtained the differential equation

$$EI\frac{d^4w}{dx^4} + b_0Cw(x) = q(x), \tag{25}$$

instead of Eq. (1). Subsequently, this equation was adapted by Schwedler, (1882) and it plays a key role in the often quoted book by Zimmermann (1887, 1930, 1941), for longitudinal-tie tracks.

The multiplication by b_0 in Eq. (24), although valid for a Winkler base that consists of closely spaced independent springs, is of questionable validity when a longitudinal-tie rests on a continuum base made up of ballast and subgrade. This was shown by Kerr (1987, pp. 39–40).

When the German and Austrian railroad engineers adapted Eq. (25) for the cross-tie track, they faced the problem of choosing the two parameters C and b_0. Since the parameter C was assumed to be independent of the tie shape, they found it necessary to establish an 'effective track width' b_0 for the cross-tie track. Saller (1932), assumed that

$$b_0 = \frac{2\ddot{u}b}{a}, \tag{26}$$

where $\ddot{u}$ is the distance from the rail center to the end of tie, b is the width of a cross-tie, as shown in Fig. 9, and a is the center-to-center tie spacing, as shown in Fig. 10.

In an attempt to prove (or justify) the validity of Saller's assumption for the determination of b_0, stated in Eq. (26), Hanker (1935) transformed the cross-tie track into a pseudo longitudinal-tie track in accordance with the scheme shown in Fig. 10. As part of this transformation, Hanker introduced a condition, that the effective tie-ballast contact areas for both cases are to be equal. Namely, that

$$ab_0 = 2\ddot{u}b. \tag{27}$$

This condition, solved for b_0, yields directly the assumption by Saller in Eq. (26).

The above transformation, and Eq. (26) for b_0, was generally accepted in the German language railroad literature. For examples refer to Hanker (1952), Schoen (1967), Eisenmann (in Fastenrath, 1981, Section 3.1), Führer (1978, Section 3.1.2) and Kaess and Gottwald (1979).

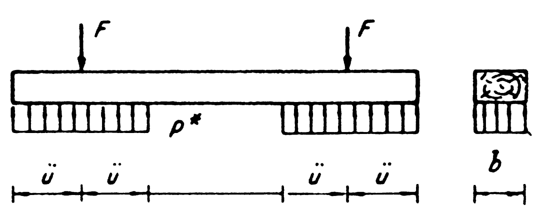

Fig. 9. The Saller assumption.

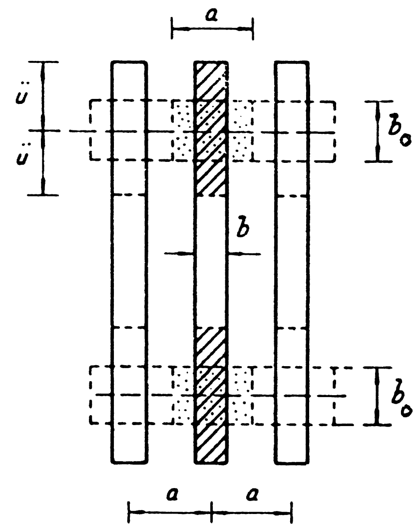

Fig. 10. Hanker transformation of cross-tie to longitudinal-tie track.

The condition of equal contact areas was apparently conceived by considerations of vertical equilibrium and by the notion that for a given rail-seat force the pressures in the effective tie-ballast contact areas should be constant and equal. Namely, that $p_0 a b_0 = p_0 2\ddot{u}b$. This is indeed the case when the rail support is represented by the Winkler base consisting of closely spaced, independent, springs. But, it is not true for an actual track base, as described previously. Therefore, the geometrical transformation shown in Fig. 10 does not correspond to an actual track situation and is of questionable validity for railroad engineering purposes.

The need to determine the 'effective track width' b_0, arose from the use of differential Eq. (25) with the a priori assumption that there exists a constant parameter C for the rail supporting base. As shown by Kerr (1987), this is not the case for actual tracks. The determination of the second parameter b_0 is also of questionable validity, as discussed above. Therefore, the use of Eq. (25) in conjunction with the two parameters b_0 and C is not justified, and hence, not advisable.

Because, of the shortcomings of the various published methods for the determination of the rail support modulus, as described above, it is suggested that for cross-tie tracks, differential Eq. (1) with the one base parameter k, be used. This parameter should be determined from one field measurement using a test car on one or two axle trucks, as presented in the beginning of this paper.

5. Problems to be considered when determining k

According to the findings by Mair (1976), Kerr and Shenton (1986) and Kerr and Eberhardt (1992), the wheel loads of the test car, P_m, should be of a magnitude anticipated in revenue service. For the standard (linear) track analyses based on Eq. (1), the corresponding

$$k = \tan\alpha = \frac{p_m}{w_m} \tag{28}$$

as indicated in Fig. 11, and is determined using Eq. (6) or graphs of the type presented in Fig. 3.

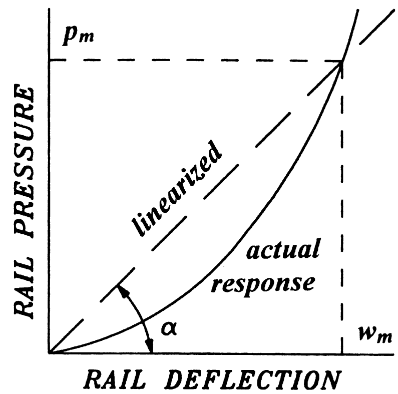

Fig. 11. Rail deflection vs. rail pressure.

This procedure yields rail bending moments that are on the safe side. However, the corresponding calculated rail seat force, F_{max}, used for the determination of the needed tie-plate size and the required depth of the ballast layer, is grossly underestimated, as shown by Kerr and Shenton (1985, 1986). The 'linear' F_{max}-value has to be multiplied by a correction factor of **1.5**, in order to represent the actual conditions in track. This was done by Kerr (1998).

Next, we consider three other problems that have to be taken into consideration when determining k. They are: (1) the effect of time-dependent rail deflections at the wheels after the test vehicle is placed on track, (2) the effect of thermal tension or compression forces in a CWR on the determined k-value using Eq. (6) or the graphs of Fig. 3 (that do not include axial forces), and (3) the effect of ballast disturbance on the rail support modulus k.

During some loading tests for the determination of k, it was observed that after placing the test vehicle on the track, in addition to the instantaneous rail deflections, the rails continued to deflect with time, especially in the vicinity of the wheel loads. In such cases the question arises as to what is w_m and when should it be recorded?

When the rails continue to deflect, this indicates that the base is visco-elastic. This is generally caused by a slow squeeze-out of the water that is trapped in a subgrade layer of poor permability (like clay or silt). For these cases the elastic springs in the Winkler model, shown in Fig. 1, have to be augmented by including viscous elements, as shown in Fig. 12.

Both models exhibit an instantaneous elastic deflection. However, in Case (a) the deflections continue for a long time (which may occur for very thick clay layers) whereas in Case (b) after a relatively short time the non-elastic deflections decrease substantially (which may occur for very thin clay layers).

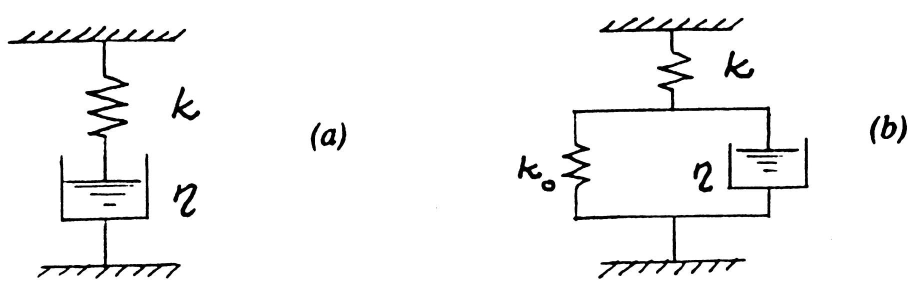

Fig. 12. Foundation models for time dependent base deflections.

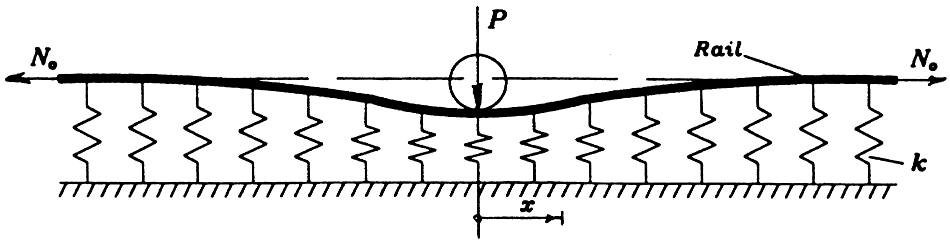

Fig. 13. Rail-in-track subjected to axial force N_0 and wheel load P.

Often, the track modulus k is needed for main line tracks, that are subjected to moving trains. In these cases there is no sufficient time for the trapped water to be squeezed out, and the track will respond elastically for both cases shown in Fig. 12. The corresponding rail support modulus k is then determined as discussed previously, by recording w_m immediately after loading and utilizing Fig. 3.

When a train stops on the track for a prolonged period of time, with a base that responds like the models in Fig. 12, then the maximum rail deflections and bending moments, hence the rail stresses, will differ from the elastic case. The analysis of these cases is more involved and requires solutions for a rail on a corresponding visco-elastic base.

The second problem to be clarified is the effect of axial tension or compression forces in the CWR,[2] caused by changes in the rail temperature, on the determination of k. To do this, consider a rail-in-track subjected to a uniform tension force N_0 and a wheel load P, as shown in Fig. 13.

The governing differential equation for this rail is

$$EIw^{IV} - N_0 w'' + kw = q \qquad -\infty < x < \infty, \tag{29}$$

where $N_0 w''$ is the term added to Eq. (1) in order to include the effect of the axial tension force. The resulting deflection at the wheel load is (Hetényi, 1947, Chapter VI)

$$w(0) = \frac{P}{2k} \frac{\sqrt{\frac{k}{4EI}}}{\sqrt{\sqrt{\frac{k}{4EI}} + \frac{N_0}{4EI}}} \tag{30}$$

Setting $w(0) = w_m$ it follows that

$$\frac{w_m}{P} = \frac{1}{2k} \frac{\sqrt{\frac{k}{4EI}}}{\sqrt{\sqrt{\frac{k}{4EI}} + \frac{N_0}{4EI}}} \tag{31}$$

When $N_0 = 0$, above equation reduces to Eq. (5), as expected. When N_0 is a compression force, N_0 is replaced by $(-N_0)$ in the above equations.

Eq. (31) was evaluated for the 115 RE rail, $N_0 = 0$ and ± 50 tons, for a range of k-values. The results are shown in Fig. 14. Compare these graphs with the ones of Fig. 3.

[2] CWR stands for Continuously Welded Rail.

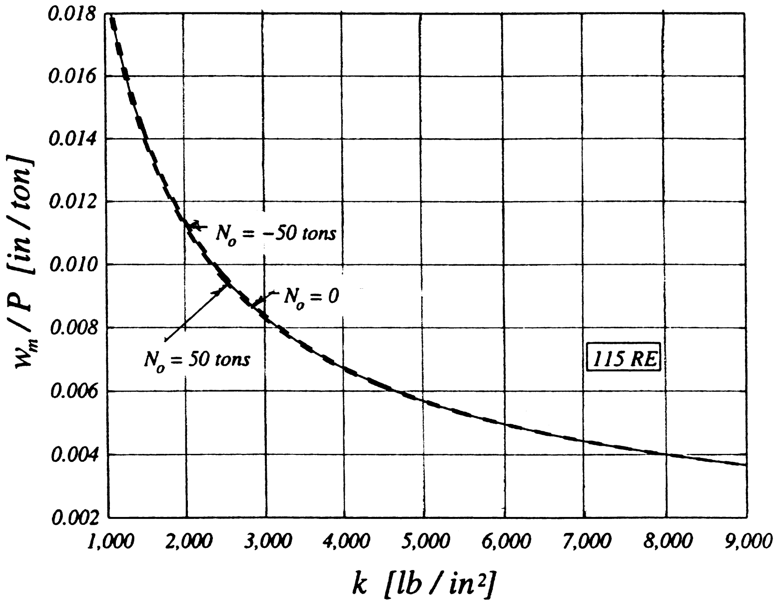

Fig. 14. Effect of N_0 on w_m/P vs. rail support modulus k.

Noting that $N_0 = 50$ tons (100,000 lb) corresponds to a temperature change from neutral[3] of about 22.2°C (40°F) , it is concluded that for the anticipated range of temperature changes, the axial force N_0 has a negligible effect on the determined k-value, using Eq. (6). This also applies to Fig. 3 for test cars with two-axle trucks. Thus, tests for the determination of the track modulus k for CWR tracks may be conducted at any reasonable ambient temperature.

In concluding this presentation, on the determination of the rail support modulus k, it should be noted that when the ballast-in-track is disturbed (for example, by tamping after timbering and surfacing or by spot renewal), the k-value drops in the respective region. Typical k-values at a track location of interest, as affected by a ballast disturbance and then by the accumulated tonnage of passing trains, is shown in Fig. 15. Note that, with increasing accumulation of tonnage after the track was tamped (or was locally disturbed), the k-value increases to a 'final' value, say to 3000 lb/in.^{2}, for a well-maintained wood-tie track. Thus, whereas a track disturbance lowers the k-value, the moving traffic tends to restore it.

This track behavior should be taken into consideration when attenpting to determine the k-value at a specific location of a railroad track.

6. Effect of dynamics on track design analyses

More than a century ago Eq. (1), $EIw^{IV} + kw = q$, was introduced for the analysis of railroad tracks. The historical evolution of the various uses of Eq. (1), as well as the controversies and related field tests to prove or disprove the various assumptions, were presented by Kerr (1976).

Since World War II, Eq. (1) was adopted as a standard for track analyses in North America, Europe, and the railways of the former Soviet Union.

[3] The *neutral temperature* in CWRs is the rail temperature at which the axial forces are zero. In North America it is generally in the range of 85°F to 115°F. It varies with the geographical location of the track territory under consideration; the high values are used in the southern part of the USA to prevent track buckling.

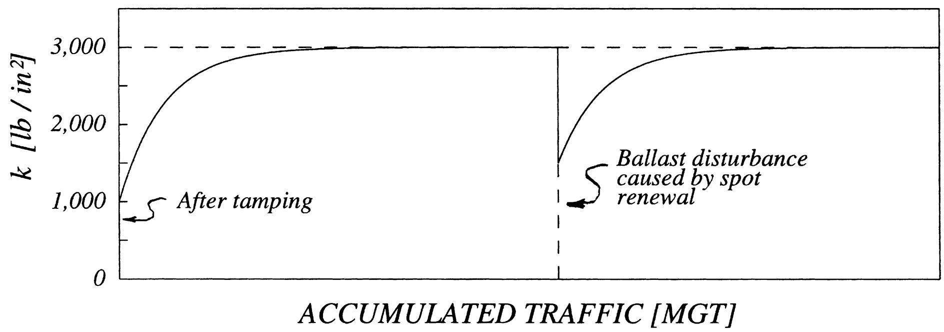

Fig. 15. Effect of ballast disturbance on the rail support modulus k of a wood-tie track at one track location.

Eq. (1) describes the static rail response. The generalization of this equation to dynamic problems encountered difficulties, in part because of the unpredictable properties of the subgrade and of ballast compaction along thousands of miles of mainline track. Also, the condition of every car in a train, as affected by wear, varies depending on the maintenance standards of the car owner. This is complicated further by the fact that the cars of one railroad often travel over the tracks of another.

Therefore, the practice that evolved throughout the world is to include the dynamic effects of the moving trains by multiplying the static wheel load by a 'speed-effect coefficient'. This coefficient is obtained from field measurements of rail strains, caused by actual trains (passenger and freight) moving in the speed range of 10–150 mph. In this approach, the rail support modulus k is determined from a static field test, as described in this paper.

For a survey of dynamic analyses of continuously supported beams subjected to moving loads, refer to Kerr (1981). For a more recent publication directly related to railway tracks, refer to Knothe and Grassie (1993).

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